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A Mathematical Approach for Describing Time-Dependent Poisson's Ratios of Periodontal Ligaments

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Abstract

Periodontal ligament is a thin layer of soft tissue that connects root of a tooth to the surrounding alveolar bone. These ligaments play an important role in initiating tooth movement when loads are applied to teeth with orthodontic appliances. The majority of such soft tissues exhibit as viscoelastic bodies or have a time-dependent behavior. Due to the viscoelastic behavior of the periodontal ligaments, the mechanical properties are strongly affected by the loading history. Poisson's ratios for the viscoelastic materials are time-dependent (in time domain) or complex frequency-dependent (in frequency domain) quantities. Moreover, three-dimensional stress fields depend on these Poisson's ratios. The main objective of this work was to develop a mathematical approach capable of determining the time-dependent Poisson's ratios of the periodontal ligaments based on experimental data of stress relaxation and creep tests. The resulting stress relaxation and creep curves are described by a three-parameter viscoelastic models. The time-dependent Poisson's ratios of the periodontal ligaments have been obtained as increasing functions of time, because shear modulus of these ligaments relaxes much more than their bulk modulus.

Keywords

Poisson's Ratio; Viscoelastic; Stress-relaxation; Creep; Periodontal ligament

Introduction

The majority of the tissues in human body such as ligaments and tendons exhibit viscoelastic or have time-dependent behavior [1-3]. The creep and relaxation are important components of tissue behavior, and investigation of such behavior takes the careful consideration. When these tissues are held at a constant strain level, stress in the tissues decreases—a phenomenon called "stress relaxation." Conversely, when held at a constant stress level, strain in the tissues increases, a condition known as "creep" [4–5]. An optimal experiment extracts the maximum amount of the useful information from the specimen being tested. This may often require performing multiple phases in the experiment, such as testing at the various strain levels or strain rates, to robustly capture the true behavior. In relaxation tests, the specimen is subjected to a certain amount of strain, and so ensuing stress is measured as a function of time. In a creep test, the specimen is subjected to a constant stress, and then the corresponding strain is measured as a function of time. Periodontal ligaments play an important role in initiating tooth movement when loads are applied to teeth with orthodontic appliances. It is also the most accessible ligament in human body as it can be directly manipulated without any surgical intervention. Peri-

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odontal ligament is a thin layer of soft tissue that connects root of a tooth to the surrounding alveolar bone. From mechanical point of view, it can be considered as a thin interface made by a solid phase, consisting mainly of collagen fibers, immersed into a so-called "ground substance."

In general, Poisson's ratio of viscoelastic tissues is a time-dependent (in time domain) or a complex frequency-dependent (in frequency domain) quantity [6, 7]. When a material is stretched, it normally not only gets longer but also contracts in the perpendicular direction. Poisson's ratio is the negative ratio of the strain in transverse direction to the strain in longitudinal direction. Since there are three dimensions for any tissue, it follows that there are six Poisson's ratios. Since also, it is common for a piece of material stretched in one direction to contract in one or both of the other directions, one of the strains is given a negative sign, thus the Poisson's ratio is usually positive.

Most material handbooks state that the Poisson's ratio can vary only between zero (or very small) and one. However, this is not true, especially for the biological materials, which are usually much more subtle than the most solid materials. The values of Poisson's ratios are influenced by the tissue structure. When you allow the constitutive holes and have the cellular textures, like the sponges, all sorts of new ways of deforming behavior become available. Most of the biological materials are expanded sideways when stretched, and therefore have an auxetic structural behavior with different Poisson's ratios.

Poisson's ratio in linear viscoelasticity is employed in the calculation of stress and strain distributions when those are expressed in terms of a modulus and a Poisson's ratio. Also the three-dimensional stress fields such as those associated with stress concentration depend on the Poisson's ratio. A time-dependent Poisson's ratio for viscoelastic tissues is associated with time-dependent stress and deformation, thus the stress concentration factors and interface stresses would depend on the time and frequency. A viscoelastic Poisson's ratio may be defined in several ways. Some of the authors have expressed concerns over some definitions of time-dependent Poisson's ratio. For example, the Poisson's ratio has been considered as ratio of the time-dependent transverse to longitudinal strain in axial extension or compression, provided that one recognizes distinction between creep and relaxation. As scientific applications, the bulk modulus for isotropic materials can be derived from the Poisson's ratio and the shear or Young's modulus, which are easier to obtain completely. Due to difficulty of measuring either the elastic or the viscoelastic bulk modulus, such implication is of high interest [7].

However, because of the nature of interrelation constitutive equations, high precision in input measurements is required. Experimentally, the Poisson's ratio can be directly determined from the measured axial and transverse strains; for isotropic materials, it can be derived from time-dependent Young's and shear moduli. Experimental implications of Poisson's ratios from direct or indirect data require a high level of accuracy [8].

Kugler, *et al*, [9] used optical methods such as laser shadow casting, or speckle interferometer for determining the Poisson's ratio of elastomeric materials. Migwi, *et al*, [10] found out that Poisson's ratios depend on the temperature, and concluded that the input viscoelastic functions should be measured in the same specimen, under same conditions, at same time, and with high precision. Lu, *et al*, [11] stated that Poisson's ratio must be determined to four significant digits to deduce the bulk modulus. Pritz [12] determined typical characters of frequency dependences of all complex moduli and complex Poisson's ratios of real solid materials by transforming the causal and real relaxation and creep responses, respectively. Tscharnuter, *et al*, [13] compared the potential of different testing methods to determine the

Poisson's ratio of polypropylene by measuring the transverse and axial strain in relaxation tests, creep experiments and displacement rate controlled tensile tests. Moreover the viscoelastic behavior has been observed and studied in articular cartilages and ligaments [14-17]. The first work for modeling of material behavior of the periodontal ligaments was based on an obvious simplification, adopting isotropic and linear elastic constitutive models to interpret its mechanical response. The mechanical properties of the periodontal ligaments have also been described earlier [17-22].

We conducted this study to determine the time-dependent Poisson's ratios of soft tissues as mechanical attributes, resistance to mechanical injury as a result of both quasistatic or dynamic loading, and constitutive force-deformation behavior of ones, such as ligaments and tendons. A novel mathematical approach was used to model the time-dependent Poisson's ratios in creep and relaxation. A generalized Wiechert model was used to model constitutive equations of viscoelastic ligaments, in which relaxation or creep function is represented by sum of a series of decaying exponential functions of time. Interrelations are shown between Poisson's ratios in relaxation and creep. Results on Poisson's ratios from confined compression tests are finally presented, where an approach was used to derive the Poisson's values from bulk and shear relaxation or creep moduli.

Mathematical Approach

The passive tensile ligament tissues are composed largely of water and collagen; they also contain very little of the proteoglycans that give cartilage its unique mechanical properties. In keeping with the functional role of these tissues, the collagen fibrils are organized primarily in long strands parallel to the axis of loading. The collagen fibrils, which may be hollow tubes, combine in a hierarchical structure, with the 20–40 nm fibrils being bundled into 0.2–12 μm fibers [2]. These fibers are birefringent under polarized light, reflecting an underlying wave or crimp structure with a periodicity between 20 and 100 μm. The fibers are bundled into fascicles, supported by fibroblasts or tenocytes, and surrounded by a fascicular membrane. Finally, multiple fascicles are bundled into a complete ligament encased in a reticular membrane. Individual collagen fibrils also display some inherent viscoelasticity, and this feature is considered to determine the viscoelastic properties of passive tensile tissues.

The Poisson's ratio is named after the French physicist Siméon Denis Poisson and is defined as the absolute value of the ratio of the transverse strain to the corresponding axial strain resulting from uniformly distributed axial stress below the proportional limit. In addition, the relation between the shear and Young's modulus of a tissue depends on the extent of volume change during the uniaxial compression or extension, which expressed as Poisson's ratio. The conventional compression tests of intact biological tissues provide an objective method for determining of the mechanical properties significant in quality evaluation and control and maximum allowable static load for minimizing mechanical damage. The results apply to quasistatic loading rather than impact loading. The most of biological tissues exhibit nonlinear behavior at larger deformations for example 0.05 strain or greater [1, 2]. Furthermore, for the completely characterizing of these materials, study of their viscoelastic behavior is needed. There are three different methods to study viscoelastic tissues—stress relaxation test, creep test, and dynamic test. The relaxation and creep tests provide the most direct technique to obtain the mechanical properties involved in linear and nonlinear theory of viscoelasticity. These tests can be conducted in the uniaxial tension or compression.

Relaxation Modeling

The viscoelastic behavior can be considered

as a spectrum with the elastic deformation on one side of the spectrum and the viscous flow on the other side of the spectrum, with varying combinations of the two spread over the range between them. Now the materials are assumed to be isotropic and linearly viscoelastic media for modeling of constitutive behavior. Relaxation and creep tests are usually applied to uniaxial deformation conditions using the following appropriate constitutive equations:

$$\sigma(t) = \int_{-\infty}^{t} E(t-\tau)\frac{d\varepsilon(\tau)}{d\tau}d\tau \quad (1)$$

$$\varepsilon(t) = \int_{-\infty}^{t} L(t-\tau)\frac{d\sigma(\tau)}{d\tau}d\tau \quad (2)$$

where, $E(t)$ and $L(t)$ are the respective uniaxial relaxation and creep functions, and τ is the integration variable. Also, σ and ε are the total stress and strain states, respectively.

Now, the uniaxial tension is considered in an elastic tissue specimen having the elastic modulus E and Poisson's ratio v. Therefore, the longitudinal strain for such cases is:

$$\varepsilon_L = \sigma_L / E \quad (3)$$

It can be assumed that fluid pressure serves a smaller role in these tissues loaded in tension, and that the complication of the biphasic model is generally unnecessary. For modeling of the segmental mechanics, it is frequently sufficient to treat these structures according to a one-dimensional approximation.

The transverse strain ε_T in terms of Poisson's ratio is:

$$\varepsilon_T = -\nu\,\sigma_L / E \quad (4)$$

The Poisson's ratio for an isotropic elastic solid in terms of the bulk modulus K or the bulk compliance $\kappa = 1/K$ can be written as follows:

$$\nu = \tfrac{1}{2} - \frac{E}{6K} = \tfrac{1}{2} - \tfrac{1}{6}\kappa E \quad (5)$$

By using Equations 4 and 5, the transverse strain can be rewritten as:

$$\varepsilon_T = -\left\langle \tfrac{1}{2} - \tfrac{1}{6}\kappa E \right\rangle \varepsilon_L \quad (6)$$

This relation between strains and moduli for elastic tissues can be converted to viscoelastic ones by the use of correspondence principle [4, 5]. At first, for simplicity of analysis, it is assumed that the bulk compliance κ is constant over the time domain. By applying correspondence principle to Equation 6, we have:

$$\bar{\varepsilon}_T(s) = -\left\langle \tfrac{1}{2} - \tfrac{1}{6}[\kappa s]\bar{E}(s) \right\rangle \bar{\varepsilon}_L(s) \quad (7)$$

By transforming back to the time domain via convolution theorem and by using derivative theorem of Laplace transform [23], the following solution is obtained:

$$\varepsilon_T(t) = -\tfrac{1}{2}\varepsilon_L(t) + \tfrac{1}{6}\kappa\int_0^t E(t-\tau)\frac{\mathrm{d}\varepsilon_L(\tau)}{\mathrm{d}\tau}\mathrm{d}\tau \quad (8)$$

This equation gives the transverse strain for any time history of longitudinal strain in terms of the constant bulk modulus κ and relaxation function E in tension or compression.

In stress relaxation test, the longitudinal strain is assumed to be $\varepsilon_L(t) = \varepsilon_0 H(t)$, in which $H(t)$ is a Heaviside step function of time (t). Therefore, by substituting it in Equation 8, we have:

$$\nu_r(t) = -\varepsilon_T(t) / \varepsilon_0 = \tfrac{1}{2} - \tfrac{1}{6}\kappa E(t) \quad (9)$$

This ratio is called a Poisson's ratio in relaxation, which is written as $\nu_r(t)$. It can be observed that $\nu_r(t)$ is less time-dependent than $E(t)$, since for a typical value $\nu = 0.3$, the second term in Equation 9 is about half the total.

Currently, if the bulk relaxation is considered as a function of time, Equation 7 can be changed into:

$$\bar{\varepsilon}_T(s) = -\left\langle \tfrac{1}{2} - \tfrac{1}{6}[s\bar{\kappa}(s)][s\bar{E}(s)] \right\rangle \bar{\varepsilon}_L(s) \quad (10)$$

For facilitating the inverse transformation

for Equation 10, using a convolution theorem, we can define:

$$[s^2\bar{E}(s)]\bar{\varepsilon}_L(s)=\bar{P}(s) \qquad (11)$$

Considering this relation, Equation 10 becomes:

$$\bar{\varepsilon}_T(s)=-\tfrac{1}{2}\bar{\varepsilon}_L(s)+\tfrac{1}{6}[\bar{\kappa}(s)][\bar{P}(s)] \qquad (12)$$

By taking an inverse transformation, done by the correspondence principle, we obtain:

$$\varepsilon_T(t)=-\tfrac{1}{2}\varepsilon_L(t)+\tfrac{1}{6}\int_0^t \kappa(t-\tau)\,P(\tau)\,\mathrm{d}\tau \qquad (13)$$

where $P(\tau)$ is determined from Equation 11 by the inverse transformation as:

$$P(\tau)=\int_0^\tau E(\tau-\eta)\frac{\mathrm{d}^2\varepsilon_L(\eta)}{\mathrm{d}\eta^2}\mathrm{d}\eta \qquad (14)$$

where η is the reduced time. Substituting Equation 14 into Equation 13, yields:

$$\varepsilon_T(t)=\tfrac{1}{6}\int_0^t \kappa(t-\tau)\int_0^\tau E(\tau-\eta)\frac{\mathrm{d}^2\varepsilon_L(\eta)}{\mathrm{d}\eta^2}\mathrm{d}\eta\,\mathrm{d}\tau-\tfrac{1}{2}\varepsilon_L \qquad (15)$$

Similarly, according to Equation 8, the relation $\varepsilon_L(t)=\varepsilon_0 H(t)$ is substituted in Equation 15. Therefore, a more general relation for Poisson's ratio in stress relaxation is obtained as:

$$\nu_r(t)=\tfrac{1}{2}-\tfrac{1}{6}\int_0^t \kappa(t-\tau)\frac{\mathrm{d}E(\tau)}{\mathrm{d}\tau}\mathrm{d}\tau \qquad (16)$$

For calculating the time-dependent Poisson's ratio of textures using Equation 16, the bulk and tensile data over the time domain are needed to be evaluated by the above convolution integral. Their evaluation is often done numerically and can lead to the time-consuming computations.

Creep Modeling

Now, the creep in uniaxial tension is considered, in which a stress of magnitude σ_0 is applied as $\sigma_L(t)=\sigma_0 H(t)$. Therefore, the longitudinal strain is time dependent as:

$$\varepsilon_L(t)=\sigma_0\,L(t)$$

By substituting the Laplace transform of Equation 17 into the master Equation 7, we have:

$$\bar{\varepsilon}_T(s)=\tfrac{1}{6}[\kappa s]\bar{E}(s)\left\langle\sigma_0\bar{L}(s)\right\rangle-\tfrac{1}{2}\bar{\varepsilon}_L(s)$$

The following relation can be resulted from the viscoelasticity theory as:

$$s^2\bar{L}(s)\bar{E}(s)=1$$

Using of this result and by applying an inverse Laplace transform to Equation 18, we have:

$$\varepsilon_T(t)=\tfrac{1}{6}\kappa\sigma_0 H(t)-\tfrac{1}{2}\varepsilon_L(t)$$

in time domain.

The Poisson's ratio in creep, *i.e.*, $\nu_c(t)$, can be finally obtained by dividing the above transverse strain by the longitudinal strain as follows:

$$\nu_c(t)=\tfrac{1}{2}-\tfrac{1}{6}\kappa\sigma_0\,{H(t)}\big/{\varepsilon_L(t)}=\tfrac{1}{2}-\tfrac{1}{6}\kappa\,{1}\big/{L(t)}$$

It can be considered that the Poisson's ratio in creep differs from the one in stress relaxation, because of $E(t)\neq 1/L(t)$.

Experimental Procedure

A developed procedure is implemented to determine the time-dependent Poisson's ratios in ligament tissues. The tissue specimens of periodontal ligaments were chosen as a case study. The samples used for mechanical measurement were carefully prepared and stored to avoid damage to specimens, and degradation of properties. The equipment to be used for mechanical measurement will depend on the technique used. The viscoelastic constitutive behavior of ligaments was modeled by a generalized Wiechert model.

The viscoelastic constitutive behaviors of the periodontal ligaments were modeled by a

general Wiechert model in relaxation and by a generalized Voigt-Kelvin model in creep. Based upon two fundamental elements of massless, Hookean linear spring and Newtonian dashpot, it is easy to construct viscoelastic models by suitable combinations of this pair of elements. A real biological tissue does not relax or retard with a single relaxation or retardation time as predicted by a Maxwell or Kelvin model. Molecular segments of varying length contribute to relaxation or retardation, with simpler and shorter segments relaxing or retarding much more quickly than long ones. This leads to a distribution of the relaxation or retardation times, which in turn produces a relaxation or retardation spread over a much longer time than could be modeled accurately with a single relaxation or retardation time. When the engineers consider incorporation of these effects is necessary, the Wiechert and generalized Voigt-Kelvin models can have as many spring-dashpot elements as needed to approximate the distribution satisfactorily. Therefore, the caused relaxation and creep moduli were represented by sum of a series of exponential terms. Schematic representations of the Wiechert and the generalized Voigt-Kelvin models are shown in Figures 1 and 2, respectively.

Viscoelastic properties of biological tissues were obtained from the stress relaxation and the creep tests on ten specimens, which independently taken from ten viscoelastic specimens of periodontal ligaments. During each test, load and displacement were both monitored continuously, which results in a load-displacement curve. A constant-rate displacement and loading histories were applied, respectively. Both load and displacement were recorded at a sampling rate of five data points per second simultaneously.

The tissue specimens were first compressed to 10% strain level at 0.5 mm/sec loading rate using an Instron universal testing machine. Then the loading rate was held at that strain level for 1000 sec. The stress relaxation curves

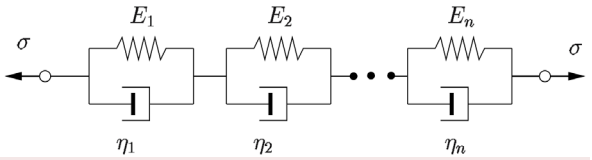

Figure 1: The biological generalized Voigt-Kelvin model

from 10 specimens were averaged, and then a Wiechert model was used to fit the average stress relaxation curve $E(t)$. As a similar way, the creep tests were performed to obtain the uniaxial creep functions $L(t)$ in linear theory of viscoelasticity.

Results and Discussion

The bulk compliance of periodontal ligaments was assumed to be constant over the time domain due to the scope of this modeling. The time-dependent creep compliance and relaxation function for tissue specimens of the periodontal ligaments were obtained by equations of analytical approach [17] and experimental data. Viscoelastic parameters such as relaxation or retardation numbers and retardation or relaxation times were determined after fitting the analytical equations into the average load-displacement curves. We found that the viscoelastic creep compliance is an increasing function of time, and the viscoelastic relaxation function is a decreasing function of time.

After determining the viscoelastic relaxation and creep moduli, the time-dependent Poisson's ratios of periodontal ligaments have been calculated by the use of Equations 16 and 21. The Poisson's ratios of periodontal liga-

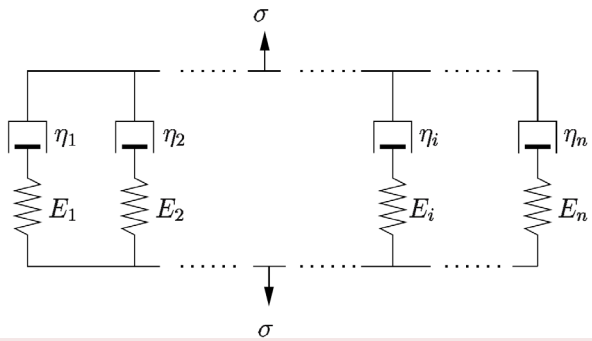

Figure 2: The biological Wiechert model

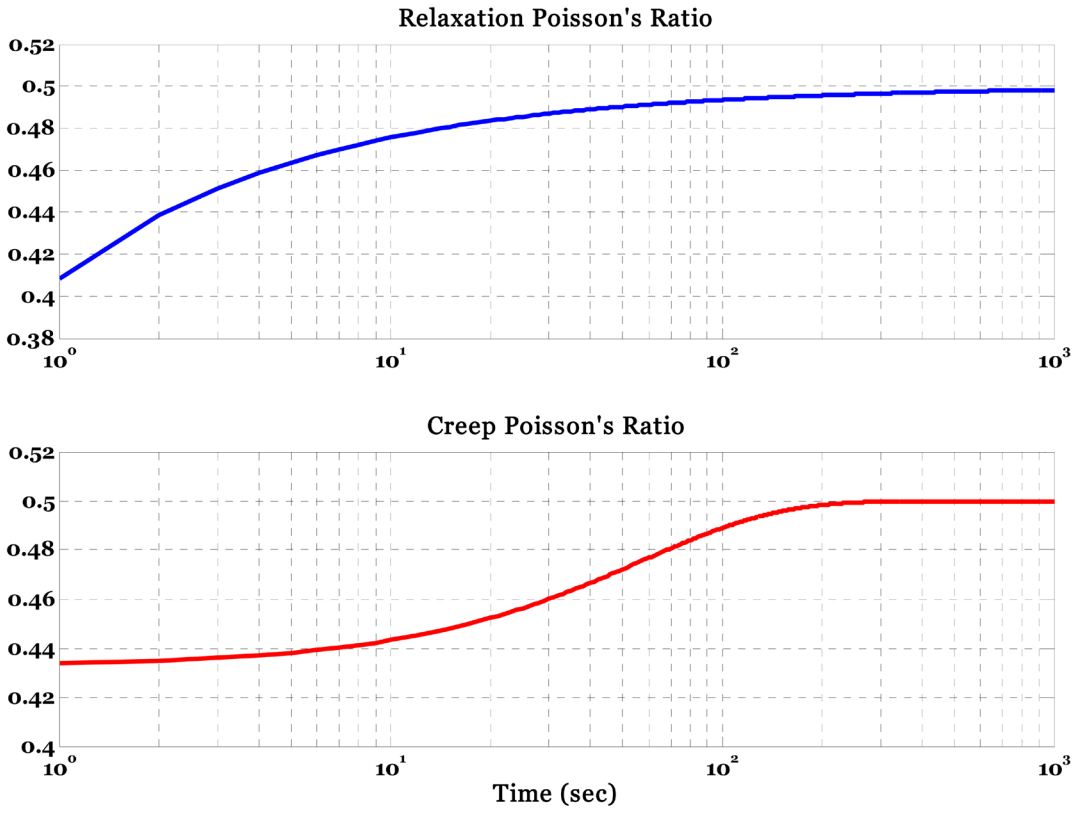

Figure 3: Poisson's ratios of periodontal ligaments in relaxation and creep

ments in relaxation and in creep are shown in Figure 3.

It can also be noted that the viscoelastic Poisson's ratios $v_c(t)$ and $v_r(t)$ of ligaments are both increasing functions of time. In addition, the time-dependent Poisson's ratio has different time dependence depending on the test procedure chosen (creep test or relaxation test). However, the difference between the Poisson's ratios in creep and in relaxation behaviors can be considered as relatively high. Moreover, considering Equations 9 and 21, we found that $v_c(t) < v_r(t)$. This result is correct considering that $E(t)\, L(t) \leq 1$.

The Poisson's ratios of viscoelastic soft tissues are time-dependent or complex frequency-dependent quantities. Three-dimensional stress fields such as those associated with stress concentration depend on the Poisson's ratio. A time-dependent Poisson's ratio for viscoelastic tissues is associated with time-dependent stress and deformation. A mathematical approach was developed capable of determining the time-dependent Poisson's ratios as mechanical characteristics of biological tissues, which are resistant to mechanical injuries as a result of both quasistatic and dynamic loadings. The Wiechert and generalized Voigt-Kelvin models were used to model the viscoelastic constitutive equations of periodontal ligaments. It was deduced that Poisson's ratios are increasing functions of time, and difference between the Poisson's ratios in creep and relaxation can be considered as relatively high.

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